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Numerical Simulation Dynamical Model of Three Species Food Chain with Holling Type-II Functional Response

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ABSTRACT

In this paper, we study an ecological model with a tritrophic food chain composed of a classical Lotka-Volterra functional response for prey and predator, and a Holling type-II functional response for predator and superpredator. There are two equilibrium points of the system. In the parameter space, there are passages from instability to stability, which are called Hopf bifurcation points. For the first equilibrium point, it is possible to find bifurcation points analytically and to prove that the system has periodic solutions around these points. Furthermore the dynamical behaviors of this model are investigated. The dynamical behavior is found to be very sensitive to parameter values as well as the parameters of the practical life. Computer simulations are carried out to explain the analytical findings.

Keywords: Food chain model, Lotka-Volterra model, Holling type-II functional response, Hopf bifurcation

INTRODUCTION

As we know that life as an ecosystem is very complex. Most of the ecological systems have the elements to produce bifurcations and dynamics behavior, and food chains are ecosystems with extremely simple structure. Modeling efforts of the dynamics of food chains which are initiated long ago, confirm that food chains have very rich dynamics. In the first place, Lotka (1925) and Volterra (1926) independently developed a simple model of interacting species that still bears their joint names which can be stated as

$$\left.\begin{aligned} \frac{dx}{dt} &= a_1 x - b_1 xy \\ \frac{dy}{dt} &= -a_2 y + b_2 xy \end{aligned}\right\} \quad (1)$$

where x is a prey, y is a predator, the predator y preys on x, a_1 is the prey growth rate in the absence of the predators, b_1 is the capture rate of prey by per predator, b_2 is the rate at which each predator converts captured prey into predator births and a_2 is the constant rate at which death in the absence of prey. They showed that ditrophic food chains (i.e. prey-predator systems) permanently oscillate for any initial condition if the prey growth rate is constant and the predator functional response is linear.

Based on experiments, Holling (1965), Hsu *et al.* (2001), Shuwen and Lansun (2005), Shuwen and Dejun (2009) suggested three different kinds of functional responses for different kinds of species to model the phenomena of predator, which resulted the standard Lotka-Volterra system more realistic. Biologically, it is quite natural for the existence and asymptotical stability of equilibria and limit cycles for autonomous predator–prey systems with these functional responses.

Almost each of the food chain models considered in ecological literature is constructed by invoking same type of functional responses for (x, y) and (y, z) populations. But a different selection of functional response would be perhaps more realistic in this context. From this point of view, we have considered a classical (nonlogistic) Lotka-Volterra functional response for the species x and y and a Holling type-II functional response for the species y and z.

MODEL SYSTEM

The classical food chain models with only two trophic levels are shown to be insufficient to produce realistic dynamics (Freedman and Waltman (1977); Hastings and Powell (1991); Hastings and Klebanoff (1993); Dubey and Upadhyay (2004)). Therefore, in this paper, we consider three species food chain interaction. With non dimensionalization, the system of three species food chain can be written as

$$\left.\begin{aligned}\frac{dx}{dt}&=\left(a_1-b_1y\right)x\\ \frac{dy}{dt}&=\left(-a_2+b_2x-\frac{c_1z}{D+y}\right)y\\ \frac{dz}{dt}&=\left(-a_3+\frac{c_2y}{D+y}\right)z\end{aligned}\right\}\tag{2}$$

where x, y, and z denote the non-dimensional population density of the prey, predator, and top predator respectively. The predator y preys on x and the predator z preys on y. Furthermore, $a_1, a_2, a_3, b_1, c_1, c_2$ and D are the intrinsic growth rate of the prey, the death rate of the predator, the death rate of the top predator, predation rate of the predator, the conversion rate, the maximal growth rate of the predator, conversion factor, and the half saturation constant respectively.

EQUILIBRIUM POINT ANALYSIS

According to Hilborn (1994) and May (2001), the equilibrium points of (2) denoted by $E(\bar{x},\bar{y},\bar{z})$, are the zeros of its nonlinear algebraic system which can be written as

$$\left.\begin{aligned}&\left(a_1-b_1y\right)x=0\\ &\left(-a_2+b_2x-\frac{c_1z}{D+y}\right)y=0\\ &\left(-a_3+\frac{c_2y}{D+y}\right)z=0\end{aligned}\right\}.\tag{3}$$

By considering the positivity of the parameters and the unknowns, we have two positive equilibrium points given by $E_0(0,0,0)$, and $E_1(x_1, y_1, 0)$ with

$$x_1=\frac{a_2}{b_2} \text{ and } y_1=\frac{a_1}{b_1},$$

and one negative equilibrium point $E_2(0, y_2, z_2)$ with

$$y_2 = \frac{a_3 D}{c_2 - a_3} \text{ and } z_2 = -\frac{a_2 D c_2}{c_1(c_2 - a_3)}$$

is not an interior equilibrium point of the system (2).

STABILITY OF EQUILIBRIUM POINTS

The dynamical behavior of equilibrium points can be studied by computing the eigenvalues of the Jacobian matrix J of system (2) where

$$J(\bar{x},\bar{y},\bar{z}) = \begin{bmatrix} a_1 - b_1\bar{y} & -b_1\bar{x} & 0 \\ b_2\bar{y} & -a_2 + b_2\bar{x} - \dfrac{c_1 D\bar{z}}{\left(D+\bar{y}\right)^2} & -\dfrac{c_1\bar{y}}{(D+\bar{y})} \\ 0 & \dfrac{c_2 D\bar{z}}{(D+\bar{y})^2} & -a_3 + \dfrac{c_2\bar{y}}{(D+\bar{y})} \end{bmatrix} \quad (4)$$

At most, there exist two positive equilibrium points for system (2). The existence and local stability conditions of these equilibrium points are as follows.

1. The Jacobian matrix (4) at the equilibrium point E_0 (0, 0, 0), is

$$J(0,0,0) = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & -a_2 & 0 \\ 0 & 0 & -a_3 \end{bmatrix} \quad (5)$$

The eigenvalues of the Jacobian matrix (5) are $\lambda_1 = a_1$, $\lambda_2 = -a_2$, and $\lambda_3 = -a_3$. Hence, the equilibrium point E_0 is a saddle point.

2. The Jacobian matrix (4) at the equilibrium point $E_1(x_1, y_1, 0)$, is

$$J(x_1, y_1, 0) = \begin{bmatrix} 0 & -\frac{b_1 a_2}{b_2} & 0 \\ \frac{b_2 a_1}{b_1} & 0 & -\frac{c_1 a_1}{b_1\left(D + \frac{a_1}{b_1}\right)} \\ 0 & 0 & -a_3 + \frac{c_2 a_1}{b_1\left(D + \frac{a_1}{b_1}\right)} \end{bmatrix} \tag{6}$$

The eigenvalues of the Jacobian matrix (6) are

$$\left.\begin{aligned} \lambda_{1,2} &= \pm i\sqrt{a_1 a_2} \\ \lambda_3 &= \frac{a_1(c_2 - a_3) - a_3 b_1 D}{b_1 D + a_1} \end{aligned}\right\} \tag{7}$$

$E_1(x_1, y_1, 0)$ is locally and asymptotically stable if

$$a_1 c_2 < a_1 a_3 + a_3 b_1 D. \tag{8}$$

HOPF BIFURCATION POINT

When we are interested to study periodic or quasi periodic behavior of a dynamical system, we need to consider the Hopf bifurcation point. The change in the qualitative character of a solution as a control parameter is varied and is known as a *bifurcation*. The dynamical system generally (Hilborn (1994) and May (2001)) can be written as

$$\dot{v} = F(v, \mu) \tag{9}$$

where

$$v = (x, y, z), \ \ \mu = (a_1, a_2, a_3, b_1, b_2, c_1, c_2, D) \tag{10}$$

According to Hilborn (1994) and May (2001), the system (2) can be written in the form (9)-(10), if an ordered pair (v_0, μ_0) satisfies the conditions

(i) $F(v_0, \mu_0) = 0$,

(ii) $J(v, \mu)$ has two complex conjugate eigenvalues $\lambda_{1,2} = a(v, \mu) \pm ib(v, \mu)$, around (v_0, μ_0),

(iii) $a(v_0, \mu_0) = 0$, $\nabla a(v_0, \mu_0) \neq 0$, $b(v_0, \mu_0) \neq 0$, and

(iv) the third eigenvalues $\lambda_3(v_0, \mu_0) \neq 0$ then (v_0, μ_0) is called a Hopf bifurcation point.

For the system (2), the equilibrium points $E_0(0,0,0)$, and $E_1(x_1, y_1, 0)$, satisfy the condition $F(v_0, \mu_0) = 0$ and for the equilibrium point $E_1(x_1, y_1, 0)$, we have two complex conjugate eigenvalues (7) with the real part of the eigenvalues are zero. The last condition $\lambda_3(v_0, \mu_0) \neq 0$ is satisfied if

$$a_1 c_2 \neq a_1 a_3 + a_3 b_1 D . \tag{11}$$

The equations (8) and (11) are satisfied if a_3 is chosen not as

$$a_{3_0} = \frac{a_1 c_2}{a_1 + b_1 D}. \tag{12}$$

Hence, E_1 is stable for $a_3 > a_{3_0}$ and unstable for $a_3 < a_{3_0}$. The point (v_0, μ_0) which corresponds to $a_3 = a_{3_0}$, is a Hopf bifurcation point. This Hopf bifurcation states sufficient condition for the existence of periodic solutions. As one parameter is varied, the dynamics of the system change from a stable spiral to a center to unstable spiral (see Table 1).

NUMERICAL SIMULATION

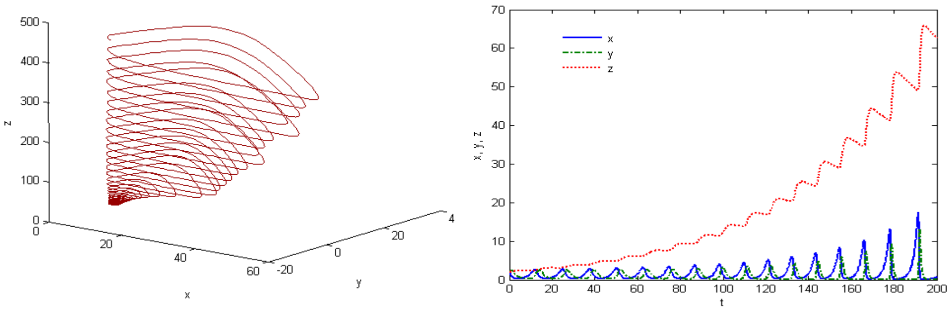
Analytical studies always remain incomplete without numerical verification of the results. In this section, we present numerical simulation to illustrate the results obtained in previous sections. The numerical simulation was implemented in MATLAB R2010a. The numerical experiments are designed to show the dynamical behavior of the system in three main different sets of parameters:

I. The case $a_3 < a_{3_0}$
II. The case $a_3 = a_{3_0}$
III. The case $a_3 > a_{3_0}$.

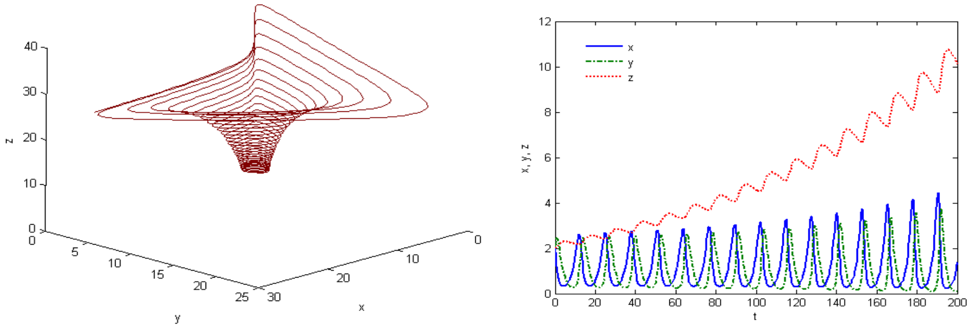
The coordinates of equilibrium points and the corresponding eigenvalues can be found in Table 1. To show the change dynamics of the system (2), the parameter set $\{a_1, a_2, b_1, b_2, c_1, c_2, D\} = \{0.5, 0.5, 0.5, 0.5, 0.6, 0.75, 25\}$ is fixed while varied. The calculation for the parameter set given Hopf bifurcation point $a_{3_0} = 0.028846$ as a reference parameter (12).

I. The case $a_3 < a_{3_0}$

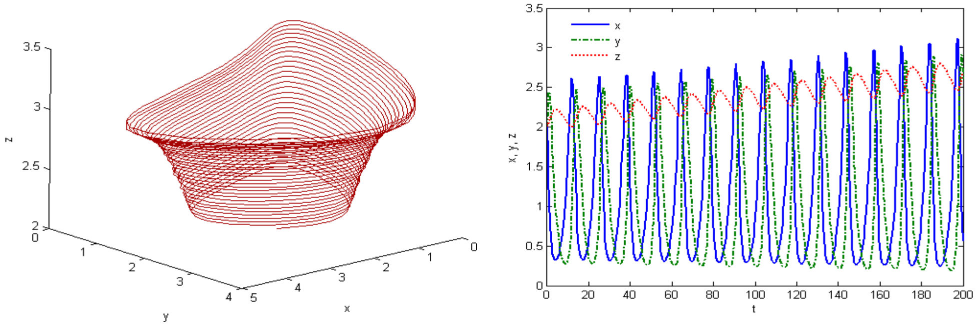
For the case $a_3 < a_{3_0}$ (see Table 1) two eigenvalues for E_1 are pure imaginary initially spiral stability corresponding to the center manifold in xy plane and one positive real eigenvalue corresponding with unstable one dimensional invariant curve in z axes. Hence the equilibrium point E_1 is a locally unstable spiral source and E_0 is a saddle point with real eigenvalues that having opposite sign. In this case the top predator z can survive, growing periodically unstable. On the other hand, prey x and predator y persist and have populations that vary periodically over time with a common period. The plot of the solution in Figure 1 exhibits this behavior.



(a) Phase space and population time series for $a_3 = 0.01$



(b) Phase space and population time series for $a_3 = 0.02$



(c) Phase space and population time series for $a_3 = 0.027$

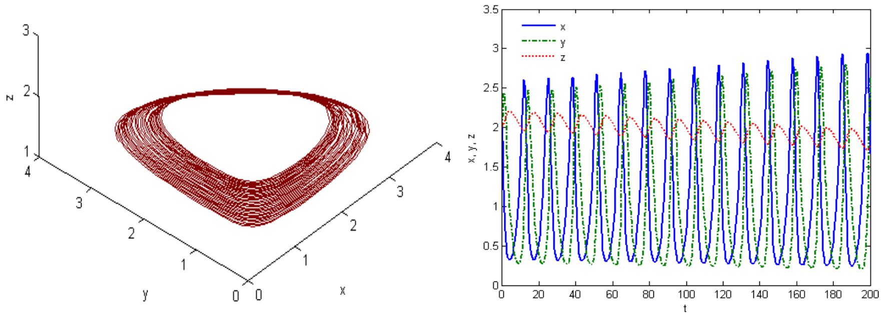
Figure 1: The solution for $a_3 < a_{3_0}$

II. The case $a_3 = a_{3_0}$

For the case $a_3 = a_{3_0}$, the equilibrium E_1 has three eigenvalues with zero real part corresponding with stable center point in xy plane (see Table 1). In this case, top predator decreases stably periodic. On the other hand prey and predator growth increases over time. The plot of the solution in Figure 2 exhibits this behavior.

TABLE 1: Numerical experiment of stability equilibrium point

The case	Parameters	Equilibrium point		Eigenvalues	
		E_0	E_1	E_0	E_1
$a_3 < a_{3_0}$	$a_3 = 0.01$	0, 0, 0	1.00000, 1.00000, 0	$\pm$ 0.50000, -0.01000	$\pm$ 0.50000 i, 0.01885
	$a_3 = 0.02$	0, 0, 0	1.00000, 1.00000, 0	$\pm$ 0.50000, -0.02000	$\pm$ 0.50000 i, 0.00885
	$a_3 = 0.027$	0, 0, 0	1.00000, 1.00000, 0	$\pm$ 0.50000, -0.02700	$\pm$ 0.50000 i, 0.0018461
$a_3 = a_{3_0}$	$a_3 = 0.028846$	0, 0, 0	1.00000, 1.00000, 0	$\pm$ 0.50000, -0.028846	$\pm$ 0.50000 i, 0
$a_3 > a_{3_0}$	$a_3 = 0.03$	0, 0, 0	1.00000, 1.00000, 0	$\pm$ 0.50000, -0.03000	$\pm$ 0.50000 i, -0.00115
	$a_3 = 0.1$	0, 0, 0	1.00000, 1.00000, 0	$\pm$ 0.50000, -0.10000	$\pm$ 0.50000 i, -0.07115
	$a_3 = 1$	0, 0, 0	1.00000, 1.00000, 0	$\pm$ 0.50000, -1.000000	$\pm$ 0.50000 i, -0.97115

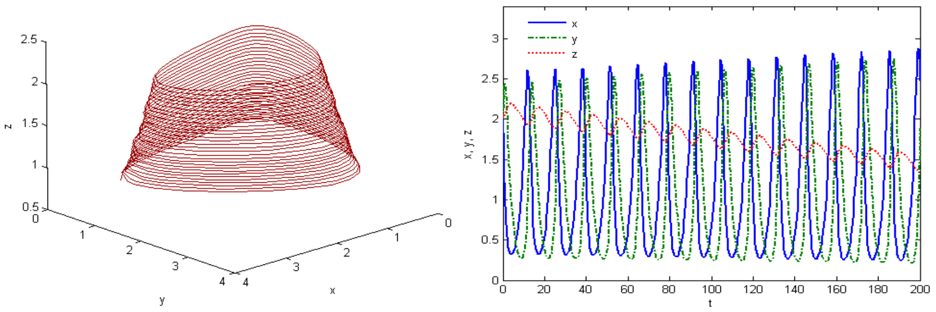


(a) Phase space for $a_{3_0} = 0.028846$ (b) Time series for $a_{3_0} = 0.028846$

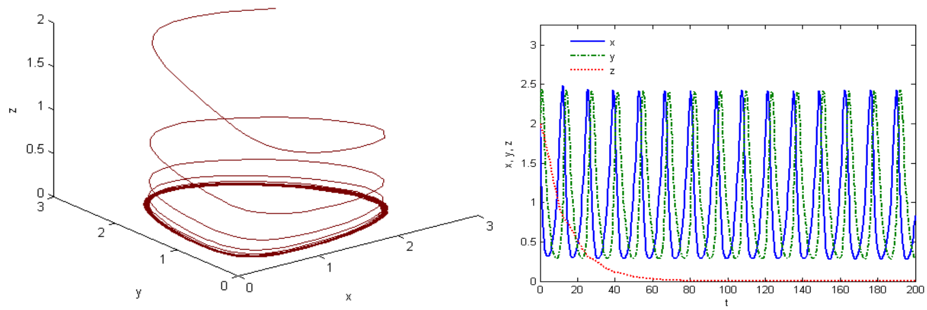
Figure 2: The solution for $a_3 = a_{3_0}$

III. The case $a_3 > a_{3_0}$

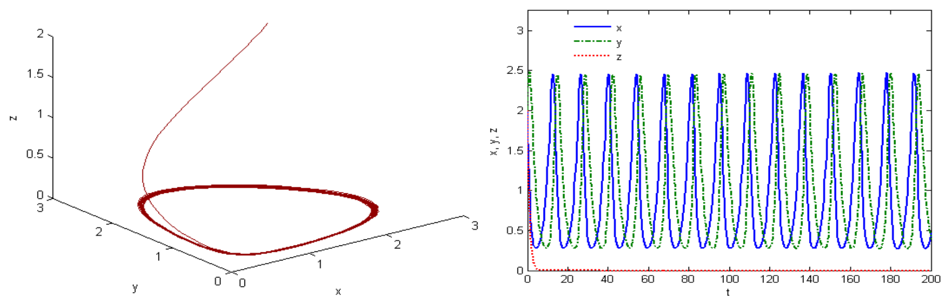
For the case $a_3 > a_{3_0}$ (see Table 1) two eigenvalues for E_1 are pure imaginary initially spiral stability corresponding to the center manifold in xy plane and one negative real eigenvalue corresponding with stable one dimensional invariant curve in z axes. Hence the equilibrium point E_1 is locally stable spiral sink and E_0 is a saddle point with real eigenvalues having opposite sign. In this case the top predator dies. On the other hand, prey x and predator y persist and have populations that vary periodically over time with a common period. The plot of the solution in Figure 3 exhibits this behavior.



(a) Phase space and population time series for $a_3 = 0.03$



(b) Phase space and population time series for $a_3 = 0.1$



(c) Phase space and population time series for $a_3 = 1$

Figure 3: The solution for $a_3 > a_{3_0}$

CONCLUSIONS

In this paper, ecological model with a tritrophic food chain composed of a classical Lotka-Volterra functional response for prey and predator, and a Holling type-II functional response for predator and superpredator is studied. In this paper, three species food chain model are analyzed and possible dynamical behavior of this system is investigated at equilibrium points. It has been shown that, the solutions possess Hopf bifurcations. The overall long-term persistence of top species z in (2) hinges solely on the parameters a_1, a_3, b_1, c_2 and *D*. In particular, if $a_3 \geq a_{3_0}$, then species z decrease over time to die out, while if $a_3 < a_{3_0}$, then species z survives. On the other hand species prey x and middle predator y can persist under all conditions. Both analytical and numerical simulations show that in certain regions of the parameter space, the model is sensitively dependent on the parameter values.

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